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AN APPROXIMATE STATICAL SOLUTION OF THE ELASTOPLASTIC INTERFACE FOR THE PROBLEM OF GALIN WITH A COHESIVE-FRICTIONAL MATERIAL

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Abstract—An approximate statically admissible solution of the elastoplastic interface is described for the plane strain problem of a pressurized circular hole in a plane subject to a non-hydrostatic stress at infinity (Problem of Galin). In contrast to the solution of Galin (*Prikl. Mat. Mekh.* **10**, 365–386 (1946)) which applies for the case of a frictionless Tresca material, it is assumed that the material is characterized by a cohesive-frictional yield strength. The solution of the elastoplastic interface is obtained in the form of a truncated series expansion, for cases where the material has yielded all around the hole. The paper discusses the limiting conditions for which the solution is applicable, and the validity of the solution in regard to an elastoplastic problem.

1. INTRODUCTION

A statically admissible solution of the elastoplastic interface, for the plane strain problem of a pressurized circular hole in an infinite medium subject to a non-hydrostatic stress at infinity, was derived by Galin[1–3] for the case of a Tresca material. The elliptic interface obtained by Galin pertains to cases where the material has yielded all around the hole and for stress boundary conditions for which the problem is statically determinate. The originality of Galin's analysis was to reduce the problem of finding the elastoplastic interface to a problem of mapping, by making use of the complex variable method[4] for solving plane elastic boundary value problems. The equivalent problem in plane stress was solved by Cherepanov[5].

The object of this paper is to extend Galin's solution for a material characterized by a Mohr–Coulomb yield strength. Both cases of passive and active† limit equilibrium in the yield zone are considered which correspond to cases where the hole is respectively expanding and contracting. The paper first describes a new approach for solving the elastoplastic interface, that leads to the formulation of a functional equation for the mapping function. It is then shown how to solve the functional equation approximately for the general case of a Mohr–Coulomb material, by seeking a solution to the mapping function in the form of a truncated series expansion. Finally, the limits of validity of the statical solution are discussed.

2. PROBLEM STATEMENT AND ASSUMPTIONS

An infinite plane with a circular hole of radius a is subject, under conditions of plane strain, to a uniform stress $\boldsymbol{\tau}^0$ at infinity and to a pressure p at the hole boundary (Fig. 1). The principal directions of the stress tensor $\boldsymbol{\tau}^0$ are parallel to the axes of the Cartesian coordinates system defined with its origin at the center of the hole. It is assumed that $\tau^0_{yy} \leqslant \tau^0_{xx} \leqslant 0$ (tension taken positive). The plane has homogeneous and isotropic properties. The material is elastic perfectly plastic and is characterized by the linear Mohr–Coulomb yield criterion

$$F := K_p\tau_1 - \tau_3 - q = 0 \tag{1}$$

† The terms active and passive are used in soil mechanics, within the context of the classical earth pressure theories of Coulomb and Rankine, to describe the loads acting on a retaining wall. There is active pressure when the backfill is pushing on the wall; passive pressure when the soil resists movement of the wall towards it.

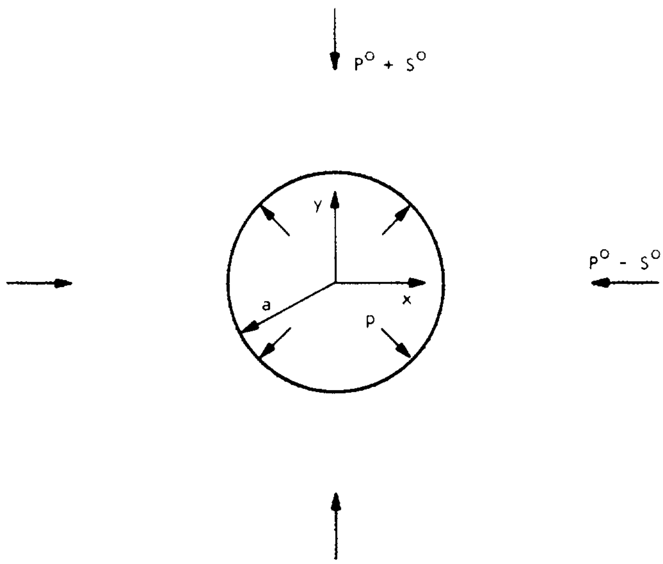

Fig. 1. Problem definition.

where K_p is the passive coefficient, function of the friction angle Φ $[K_p = (1 + \sin\Phi)/(1 - \sin\Phi)]$, and q is the unconfined compressive strength ($q = 2c\sqrt{K_p}$, where c is cohesion).

Poisson's ratio ν of the material is restricted to obey the inequality

$$\nu \geqslant \frac{1}{(K_p + 1)} \tag{2}$$

which, as proven in Appendix A, represents a sufficient condition for the out-of-plane stress τ_{zz} to be the strict intermediate principal stress, in regions of plastic flow. (This restriction reduces to the incompressibility condition for a cohesive frictionless material.) Under inequality (2), the Mohr–Coulomb criterion can always be expressed in terms of the plane components of the stress tensor; in which case a useful form of the criterion is in terms of P and S, the mean pressure and the stress deviatoric invariants in the plane of the deformation

$$S = S_l(P) \tag{3}$$

where

$$P = -\frac{(\tau_{xx} + \tau_{yy})}{2}; \qquad S = \left[\left(\frac{\tau_{yy} - \tau_{xx}}{2}\right)^2 + \tau_{xy}\right]^{1/2} \tag{4}$$

and S_l is the yield limit, function of P, of the invariant S

$$S_l = \frac{K_p - 1}{K_p + 1}\left(P + \frac{q}{K_p - 1}\right). \tag{5}$$

The objective of this paper is to derive a statical solution of the elastoplastic interface, for combinations of τ^0 and p for which the hole is completely surrounded by a plastic zone. (The formation of the yield zone is assumed to be the result of monotonic loading.)

The construction of the statical solution is based on two *a priori* hypotheses.

(1) There is no elastic unloading taking place in regions having experienced plastic deformation.

(2) The problem remains statically determinate, since the inception of the first plastic zone.

These assumptions impose various restrictions on the parameters controlling the problem, that will be examined in Section 6.

3. SOLUTION OF THE INTERFACE AS A PROBLEM OF MAPPING

By postulating no elastic unloading in regions of plastic flow, in addition to statical determinacy, a statical solution of the interface, corresponding to an internal pressure p in the hole and a mean pressure P^0 and a stress deviatoric S^0 at infinity, can be constructed while ignoring the sequence of loading. These two *a priori* assumptions imply that the plastic zone develops adjacent to the circular boundary and that the plastic stress field is completely determined by the internal pressure p[6]. The location of the elastoplastic interface and the elastic stress field can then be computed from the stress condition at infinity and the requirement of stress continuity at the interface[7].

3.1. *Plastic stress field*

The postulates of statical determinacy and no elastic unloading imply that the stress field in the plastic zone is axisymmetric. Two solutions exist, passive and active, corresponding respectively to $\tau_{\phi\phi} > \tau_{rr}$ and $\tau_{\phi\phi} < \tau_{rr}$. In the elastoplastic problem with a hydrostatic stress at infinity ($S^0 = 0$), the passive solution applies for cases where $p > P^0$, the active one for $p < P^0$. For both modes of failure, the plastic stress field is given by the general formula[8,9].

$$\tau_{rr} = \frac{q}{K_p - 1} - \left(p + \frac{q}{K_p - 1}\right)\left(\frac{r}{a}\right)^{K-1}$$

$$\tau_{\phi\phi} = \frac{q}{K_p - 1} - K\left(p + \frac{q}{K_p - 1}\right)\left(\frac{r}{a}\right)^{K-1}$$

$$\tau_{r\phi} = 0 \tag{6}$$

where (r, ϕ) are the cylindrical coordinates of a point of the plastic region, and where K is to be replaced by K_p for an active yield zone and by $K_a = 1/K_p$ for a passive one.

3.2. *Stresses at the elastoplastic interface*

Let S^- designate the infinite elastic region bounded by the unknown interface Γ. The complex variable $Z = x + iy$ is defined in the physical plane (also referred to as the Z-plane). Consider then the mapping function

$$Z = \omega(\zeta) \tag{7}$$

which maps conformally Σ^-, the region exterior to the unit circle γ centered at the origin of a reference ζ-plane, onto S^- such that there is correspondence between the point at infinity and the positive real axis in the two regions (Fig. 2). Since the coordinate axes are parallel to the principal directions of the stress at infinity, the elastoplastic interface is necessarily symmetric with respect to the x- and y-coordinate axes. It follows, from these symmetries, and the particular choice of the mapping function that

$$\omega(\zeta) = -\omega(-\zeta) \quad \text{and} \quad \omega(\zeta) = \bar{\omega}(\zeta) \tag{8}$$

where $\bar{\omega}(\zeta)$ stands for $\overline{\omega(\bar{\zeta})}$, following Muskhelishvili's notation[4]. Near the point at infinity, $\omega(\zeta)$ behaves as $R\zeta$, where R is a real coefficient.

Let $r_*(\sigma)$, where σ is a point on γ, designate the distance between the origin of the Z-plane to a point $Z_* = \omega(\sigma)$ on the interface Γ

$$r_*(\sigma) = [\omega(\sigma)\overline{\omega(\sigma)}]^{1/2}. \tag{9}$$

Using the plastic stress solution, eqns (6) and (9), and coordinates transformation formula, the plane Cartesian components of the stress tensor along Γ are then given by

$$\begin{aligned} \frac{1}{2}(\tau_{xx} + \tau_{yy}) &= \frac{q}{K_p - 1} - \frac{K+1}{2}\left(p + \frac{q}{K_p - 1}\right)\left[\frac{r_*(\sigma)}{a}\right]^{K-1} \\ \frac{1}{2}(\tau_{yy} - \tau_{xx}) + \mathrm{i}\,\tau_{xy} &= -\frac{K-1}{2}\left(p + \frac{q}{K_p - 1}\right)\left[\frac{r_*(\sigma)}{a}\right]^{K-1}\frac{\overline{\omega(\sigma)}}{\omega(\sigma)}. \end{aligned} \tag{10}$$

The above equations can be rewritten using two new quantities R_0 and S_1^0 respectively defined as

$$R_0 = \left[\frac{2}{K+1}\,\frac{P^0 + \dfrac{q}{K_p - 1}}{p + \dfrac{q}{K_p - 1}}\right]^{1/(K-1)} \tag{11}$$

$$S_1^0 := S_1(P^0) = \frac{K_p - 1}{K_p + 1}\left(P^0 + \frac{q}{K_p - 1}\right). \tag{12}$$

Here R_0 represents the radius of the elastoplastic interface, normalized with respect to a, in the particular case of a hydrostatic stress at infinity[8,9] and S_1^0 is the yield limit of the stress deviatoric invariant at infinity. Using eqns (11) and (12), eqn (10) becomes

$$\begin{aligned} \frac{1}{2}(\tau_{xx} + \tau_{yy}) &= \frac{q}{K_p - 1} - \frac{K_p + 1}{K_p - 1}S_1^0\,\hat{r}_*^{K-1}(\sigma) \\ \frac{1}{2}(\tau_{yy} - \tau_{xx}) + \mathrm{i}\,\tau_{xy} &= \mp S_1^0\,\frac{\overline{\hat{\omega}(\sigma)}}{\hat{\omega}(\sigma)}\,\hat{r}_*^{K-1}(\sigma) \end{aligned} \tag{13}$$

where

$$\hat{r}_*(\sigma) = \frac{r_*(\sigma)}{aR_0}; \qquad \hat{\omega}(\sigma) = \frac{\omega(\sigma)}{aR_0}.$$

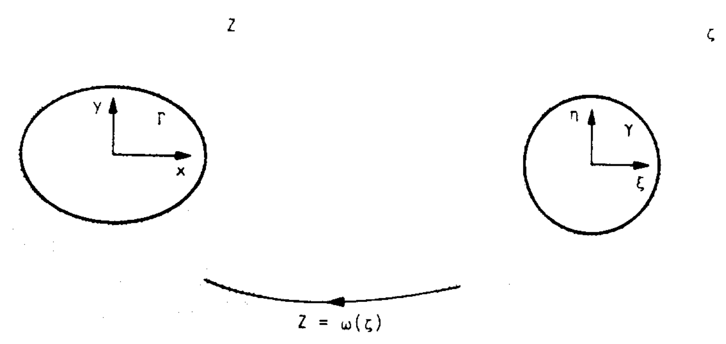

Fig. 2. Conformal mapping.

In eqn $(13)_2$ the upper and lower signs refer to the active ($K = K_p$) and passive ($K = K_a$) modes, respectively. (This convention will apply to all equations with a double sign.) It can be seen from eqns (13), that the stresses at the elastoplastic interface becomes independent of the internal pressure p with the transformation $Z' = Z/aR_0$. The Z'-plane is defined as the unit plane.

3.3. *Continuity of the stresses at the elastoplastic interface*

The stresses in the elastic region S^- can be expressed in terms of the mapping function $\omega(\zeta)$ and two functions $\phi_1(\zeta)$ and $\psi_1(\zeta)$ analytic in Σ^- [4]

$$\frac{1}{2}(\tau_{xx} + \tau_{yy}) = -P^0 + \phi_1(\zeta) + \overline{\phi_1(\zeta)}$$

$$\frac{1}{2}(\tau_{yy} - \tau_{xx}) + \mathrm{i}\,\tau_{xy} = -S^0 + \frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\phi_1'(\zeta) + \psi_1(\zeta). \tag{14}$$

The above expressions are slightly different from the usual Kolossof formulae. Functions $\phi_1(\zeta)$ and $\psi_1(\zeta)$ are actually associated to the stress difference $\boldsymbol{\tau}^1 = \boldsymbol{\tau} - \boldsymbol{\tau}^0$; they are analytic in S^- and $O(\zeta^{-2})$ near the point at infinity. The behavior at infinity can be deduced from the conditions at infinity on the stress, and the additional assumption that the rigid body rotation is zero at infinity.

Continuity of the stresses along Γ is achieved by equating the right-hand members of eqns (13) and (14), with ζ in eqns (14) set to σ

$$\phi_1(\sigma) + \overline{\phi_1(\sigma)} = S_1^0\frac{K_p + 1}{K_p - 1}[1 - \dot{r}^{K-1}(\sigma)]$$

$$\frac{\overline{\hat{\omega}(\sigma)}}{\hat{\omega}'(\sigma)}\phi_1'(\sigma) + \psi_1(\sigma) = S^0 \mp S_1^0\frac{\overline{\hat{\omega}(\sigma)}}{\hat{\omega}(\sigma)}\dot{r}^{K-1}(\sigma). \tag{15}$$

It follows from the above equations that the two analytic functions $\phi_1(\zeta)$ and $\psi_1(\zeta)$ can naturally be normalized by S_1^0

$$\hat{\phi}_1(\sigma) + \overline{\hat{\phi}_1(\sigma)} = \frac{K_p + 1}{K_p - 1}[1 - \dot{r}^{K-1}(\sigma)]$$

$$\frac{\overline{\hat{\omega}(\sigma)}}{\hat{\omega}'(\sigma)}\hat{\phi}_1'(\sigma) + \hat{\psi}_1(\sigma) = m \mp \frac{\overline{\hat{\omega}(\sigma)}}{\hat{\omega}(\sigma)}\dot{r}^{K-1}(\sigma) \tag{16}$$

where

$$\hat{\phi}_1(\zeta) = \frac{\phi_1(\zeta)}{S_1^0}; \qquad \hat{\psi}_1(\zeta) = \frac{\psi_1(\zeta)}{S_1^0}$$

and

$$m = \frac{S^0}{S_1^0}. \tag{17}$$

The positive parameter m is defined as the obliquity of the stress at infinity. In the stress invariant space (P^0, S^0), stress states characterized by the same obliquity m lay on a line which passes through the intersection of the Mohr–Coulomb envelope with the hydrostatic axis $S^0 = 0$.

A similar procedure for the case of a Tresca material ($\Phi = 0$) leads to

$$\hat{\phi}_1(\sigma) + \overline{\hat{\phi}_1(\sigma)} = 1 \mp \ln[\hat{\omega}(\sigma)\overline{\hat{\omega}(\sigma)}]$$
$$\frac{\overline{\hat{\omega}(\sigma)}}{\hat{\omega}'(\sigma)}\hat{\phi}_1'(\sigma) + \hat{\psi}_1(\sigma) = m \mp \frac{\overline{\hat{\omega}(\sigma)}}{\hat{\omega}(\sigma)}. \tag{18}$$

In this case, S_1^0 degenerates into c, the cohesion.

Equations (16) express, respectively, continuity of the mean pressure and the stress deviatoric across the elastoplastic interface Γ. Mathematically, they represent boundary conditions on γ to be satisfied by the three unknown analytic functions $\hat{\omega}(\zeta)$, $\hat{\phi}_1(\zeta)$, and $\hat{\psi}_1(\zeta)$. These boundary conditions, together with the conditions at infinity for these functions, are in principle sufficient to determine the position of the interface Γ and the elastic stress field.

4. FUNCTIONAL EQUATION FOR THE MAPPING FUNCTION

Equations for solving the mapping function $\hat{\omega}(\zeta)$ are derived in the following, together with general expressions for the two analytic functions $\hat{\phi}_1(\zeta)$ and $\hat{\psi}_1(\zeta)$. This derivation relies on the Schwarz's reflection principle and on Laurent's decomposition principle (e.g. Ref. [10]). For this particular case, these two principles can be stated as follows.

(1) *Schwarz's reflection.* If $F(\zeta)$ is analytic in Σ_1^- ($\rho > \rho_1$), then $\bar{F}(\zeta^{-1})$ is analytic in Σ_1^+ ($\rho < \rho_1^{-1}$); $F(\zeta)$ and $\bar{F}(\zeta^{-1})$ are said to be associated functions.

(2) *Laurent's decomposition.* If $G(\zeta)$ is analytic in the ring Ω ($\rho_1 < \rho < \rho_1^{-1}$), then $G(\zeta)$ can be decomposed, inside Ω, as the sum of two functions $G_1(\zeta)$, analytic in Σ_1^+, and $G_2(\zeta)$ analytic in Σ_1^- and vanishing at infinity.

Consider eqn $(16)_1$, which expresses continuity of the mean pressure across Γ. Provided that Γ is smooth, $r_*^{K-1}(\sigma)$ is continued analytically on both sides of γ by the function $r_*^{K-1}(\zeta)$ defined as

$$r_*^{K-1}(\zeta) = [\hat{\omega}(\zeta)\bar{\hat{\omega}}(\zeta^{-1})]^{(K-1)/2}. \tag{19}$$

Let ρ_1 designate the modulus of the zero or singularity of $\hat{\omega}(\zeta)$, which is the farthest away from the origin of the ζ-plane ($\rho_1 < 1$). Since the function $r_*^{K-1}(\zeta)$ is analytic inside the ring Ω ($\rho_1 < \rho < \rho_1^{-1}$), it follows from Laurent's decomposition principle that $r_*^{K-1}(\zeta)$ can be decomposed into two analytic functions. However, because of the properties, eqns (8), of $\omega(\zeta)$ and the definition of $r_*^{K-1}(\zeta)$, this decomposition takes the particularly simple form

$$r_*^{K-1}(\zeta) = \bar{d}(\zeta^{-1}) + d_0 + d(\zeta) \tag{20}$$

where d_0 is a real coefficient and $d(\zeta)$ a function analytic in Σ_1^- and vanishing at infinity.

Using eqn (20), the first boundary condition, eqn $(16)_1$ can be rewritten as

$$\hat{\phi}_1(\sigma) + \frac{K_p+1}{K_p-1}d(\sigma) = \frac{K_p+1}{K_p-1}(1-d_0) - \overline{\hat{\phi}_1(\sigma)} + \frac{K_p+1}{K_p-1}\overline{d(\sigma)} \tag{21}$$

or

$$F^-(\sigma) = F^+(\sigma) \tag{22}$$

where $F^-(\sigma)$ is the boundary value on γ of $F^-(\zeta)$ analytic in Σ^- and $F^+(\sigma)$ the boundary value of the function $F^+(\zeta)$ analytic in Σ^+ ($\rho < 1$)

$$F^-(\zeta) := \hat{\phi}_1(\zeta) + \frac{K_p+1}{K_p-1}d(\zeta)$$
$$F^+(\zeta) := \frac{K_p+1}{K_p-1}(1-d_0) - \bar{\hat{\phi}}_1(\zeta^{-1}) - \frac{K_p+1}{K_p-1}\bar{d}(\zeta^{-1}). \tag{23}$$

Consequently, the functions $F^+(\zeta)$ and $F^-(\zeta)$ are an analytic continuation of each other across the unit circle γ. But according to Liouville's theorem, $F^+(\zeta)$ and $F^-(\zeta)$ are identically equal to a constant, which is zero because of the vanishing of $F^-(\zeta)$ at infinity. It follows therefore that

$$\hat{\phi}_1(\zeta) = -\frac{K_p+1}{K_p-1}d(\zeta), \qquad \zeta \in \Sigma_1^- \tag{24}$$

which indicates that $\hat{\phi}_1(\zeta)$ is actually analytic in Σ_1^-. Using the concept of associated functions, $F^+(\zeta)$ can be written as

$$F^+(\zeta) = \frac{K_p+1}{K_p-1}(1-d_0) - \bar{F}^-(\zeta^{-1}). \tag{25}$$

Since $F^+(\zeta) = \bar{F}^-(\zeta^{-1}) = 0$, expression (25) shows that

$$d_0 = 1. \tag{26}$$

Equation (26) represents a scalar constraint that must be satisfied by the mapping function $\hat{\omega}(\zeta)$.

Consider next the second boundary condition, eqn (16), which expresses continuity of the deviatoric stress across the elastoplastic interface. Taking into account eqn (24), and after multiplication by $\hat{\omega}'(\sigma)$, this boundary condition becomes

$$-\frac{K_p+1}{K_p-1}\overline{\hat{\omega}(\sigma)}\,d'(\sigma) + \hat{\omega}'(\sigma)\hat{\psi}_1(\sigma) = m\hat{\omega}'(\sigma) \mp \hat{\omega}'(\sigma)\frac{\overline{\hat{\omega}(\sigma)}}{\hat{\omega}(\sigma)}f_*^{K-1}(\sigma). \tag{27}$$

The first term on the left-hand side of eqn (27) is the value on the unit circle γ of the function $g(\zeta)$

$$g(\zeta) = -\frac{K_p+1}{K_p-1}\bar{\hat{\omega}}(\zeta^{-1})d'(\zeta) \tag{28}$$

which is analytic in Ω (note that $\bar{\hat{\omega}}(\zeta^{-1})$ is analytic in Σ_1^+, $d'(\zeta)$ in Σ_1^-). Using Laurent's decomposition, $g(\zeta)$ can be decomposed inside Ω, into $g_1(\zeta)$ analytic in Σ_1^+, and $g_2(\zeta)$ analytic in Σ_1^-, and vanishing at infinity

$$g(\zeta) = g_1(\zeta) + g_2(\zeta). \tag{29}$$

The second term on the left-hand side of eqn (27) is the boundary value of $\hat{\omega}'(\zeta)\hat{\psi}_1(\zeta)$ which is analytic in Σ_1^- and is $O(\zeta^{-2})$ near the point at infinity.

The first term on the right-hand side of eqn (27) corresponds to $m\hat{\omega}'(\zeta)$ which is analytic in Σ_1^-, but equal to $m\lambda$ at infinity where λ is defined as R/aR_0. The second term is the value on γ of the function $h(\zeta)$ analytic in Ω

$$h(\zeta) := \hat{\omega}'(\zeta)\frac{\bar{\hat{\omega}}(\zeta^{-1})}{\hat{\omega}(\zeta)}f_*^{K-1}(\zeta) \tag{30}$$

and thus $h(\zeta)$ can be equated in Ω to the sum of $h_1(\zeta)$ and $h_2(\zeta)$, $h_1(\zeta)$ analytic in Σ_1^+ and $h_2(\zeta)$ analytic in Σ_1^- and vanishing at infinity.

In light of the decomposition, continuity of the stress deviatoric, eqn (27), across Γ can be rewritten as

$$G^-(\sigma) = G^+(\sigma) \tag{31}$$

where $G^-(\sigma)$ is the boundary value of $G^-(\zeta)$ analytic in Σ^- and vanishing at infinity and $G^+(\zeta)$ is a function analytic in Σ^+

$$
\begin{aligned}
G^-(\zeta) &= \hat{\omega}'(\zeta)\hat{\psi}_1(\zeta) - m\hat{\omega}'(\zeta) + m\lambda + g_2(\zeta) \pm h_2(\zeta) \\
G^+(\zeta) &= -g_1(\zeta) \mp h_1(\zeta) + m\lambda.
\end{aligned} \tag{32}
$$

Then pursuing the argument developed previously, it is found that

$$\hat{\psi}_1(\zeta) = [m\hat{\omega}'(\zeta) - m\lambda - g_2(\zeta) \mp h_2(\zeta)]/\hat{\omega}'(\zeta), \qquad \zeta \in \Sigma^- + \gamma \tag{33}$$

$$G^+(\zeta) = 0, \qquad \zeta \in \Sigma_1^+. \tag{34}$$

Equation (33) provides a means of calculating $\psi_1(\zeta)$ once the mapping function $\hat{\omega}(\zeta)$ has been determined. Equation (34) represents a functional constraint for $\hat{\omega}(\zeta)$.

The mapping function $\hat{\omega}(\zeta)$ can now be determined from eqn (26) and from the functional eqn (34). The procedure is complicated, however, because Laurent's decomposition of the functions $g(\zeta)$, $h(\zeta)$, and $d(\zeta)$ can, in general, only be accomplished by means of Laurent's series representations of these functions.

The next section describes a method of obtaining an estimate $\hat{\omega}^{(n)}(\zeta)$ to $\hat{\omega}(\zeta)$, by solving eqn (34) approximately. It should be noted, however, that for the case of a Tresca material, the procedure described above provides a direct and simple means of calculating $\hat{\omega}(\zeta)$ (Galin's solution), and the analytic function $\hat{\phi}_1(\zeta)$ and $\hat{\psi}_1(\zeta)$

$$
\begin{aligned}
F^+(\zeta) = 0: &\quad \lambda = 1 \\
G^+(\zeta) = 0: &\quad \hat{\omega}(\zeta) = \zeta \pm \frac{m}{\zeta} \\
F^-(\zeta) = 0: &\quad \hat{\phi}_1(\zeta) = \mp \ln \frac{\hat{\omega}(\zeta)}{\zeta} \\
G^-(\zeta) = 0: &\quad \hat{\psi}_1(\zeta) = \frac{m^2+1}{m \mp \zeta^2}.
\end{aligned} \tag{35}
$$

It is emphasized that the approach described in this paper is different from the method of Galin[1], and Cherepanov[5]. While Galin's approach makes use of the biharmonicity of the plastic stress function in the particular case of a Tresca material and plane strain (only a handful of plastic states are characterized by a biharmonic stress function), the method of Cherepanov relies, among other things, on explicitly solving the second boundary condition, eqn (16), for $\overline{\hat{\omega}(\sigma)}$.

5. APPROXIMATE SOLUTION OF THE ELASTOPLASTIC INTERFACE

An approximate solution of the elastoplastic interface is now sought in the form of a truncated series expansion of $\hat{\omega}(\zeta)$. Laurent's series expansion of $\hat{\omega}(\zeta)$ with respect to $\zeta = 0$, for ζ large enough, is

$$\hat{\omega}(\zeta) = \lambda\zeta\left(1 + \sum_{j=1}^{\infty} \frac{m_{2j}}{\zeta^{2j}}\right) \tag{36}$$

where all the coefficients m_{2j} are real, like λ. The particular form of the series expansion (36) can be deduced from the properties, eqns (8), of the mapping function. The series representation of $\hat{\omega}(\zeta)$ is valid for any $\rho > \rho_*$, $\rho_*(<1)$ being the modulus of the singularity of $\hat{\omega}(\zeta)$ closest to γ.

One now seeks to calculate the approximation $\hat{\omega}^{(n)}(\zeta)$ to $\hat{\omega}(\zeta)$

$$\hat{\omega}^{(n)}(\zeta) = \lambda^{(n)}\zeta\left(1 + \sum_{j=1}^{n} \frac{m_{2j}^{(n)}}{\zeta^{2j}}\right) \tag{37}$$

by determining the coefficients $\lambda^{(n)}$, $m_{2j}^{(n)}$, $j = 1,n$ as approximation of "order n" of the coefficients λ, and m_{2j}, $j = 1,n$ of the series (36). The coefficients of $\hat{\omega}^{(n)}(\zeta)$ are calculated using constraint (26) and by satisfying approximately the functional equation $G^+(\zeta) = 0$.

5.1. *Non-linear system of equations*

Since $G^+(\zeta)$ is a function analytic in Σ_1^+, it can be expanded into a Taylor series around $\zeta = 0$. Due to the symmetries of the problem, this series takes the particular form

$$G^+(\zeta) = \sum_{j=0}^{\infty} G_{2j}^+\zeta^{2j}, \qquad \zeta \in \Sigma_1^+ \tag{38}$$

with all the coefficients G_{2j} real. The functional constraint $G^+(\zeta) = 0$ is satisfied approximately by imposing that the n first coefficients of series (38) are zero

$$G_{2j}^+ = 0, \qquad j = 0, n-1. \tag{39}$$

Together with eqn (26), the n equations (39) constitute a system of $(n + 1)$ equations in the unknown $\lambda^{(n)}$, $m_{2j}^{(n)}$, $j = 1,n$. Actually, as shown in Appendix B, this system can simply be decomposed into one equation giving $\lambda^{(n)}$ in terms of $m_{2j}^{(n)}$, $j = 1,n$

$$\lambda^{(n)} = h_0^{(n)}(m_2^{(n)}, m_4^{(n)}, \ldots, m_{2n}^{(n)}; K, m) \tag{40}$$

and into n non-linear equations in the unknowns $m_{2j}^{(n)}$, $j = 1,n$

$$h_{2j}^{(n)}(m_2^{(n)}, m_4^{(n)}, \ldots, m_{2n}^{(n)}; K, m) = 0; \qquad j = 1, n. \tag{41}$$

The explicit form of eqns (40) and (41) is derived in Appendix B.

Since K and m are the only parameters present in the non-linear systems of eqn (41), the shape of the elastoplastic interface depends only upon the obliquity m of the stress at infinity, the friction angle of the material, and the mode of failure (passive or active).

5.2. *First-order approximation of the interface*

Successful numerical resolution of the non-linear system, eqn (41), relies on a "good" initial guess of the roots $m_{2j}^{(n)}$, $j = 1,n$. These initial values are obtained by assuming the interface to be elliptic, i.e. by assigning a zero value to the initial guess of $m_4^{(n)}$, $m_6^{(n)}, \ldots$, and by calculating a first-order approximation of $m_2^{(n)}$. This first-order approximation is calculated by linearizing $h_2^{(1)}$ $(m_2^{(1)}; K, m) = 0$, thereby yielding (see Appendix B)

$$m_2 \cong \pm\frac{2}{K+1}m. \tag{42}$$

Interestingly, this first-order approximation gives the correct result for the limiting case of a frictionless material (Galin's solution).

The coefficient $\lambda^{(n)}$ is calculated from eqn (40), after solving the non-linear system of eqns (41). Note that a "first-order" approximation of λ is (see Appendix B)

$$\lambda \cong \left[1 + \left(\frac{K_p - 1}{K_p + 1}m\right)^2\right]^{1/(1-K)} \tag{43}$$

The coefficient λ is thus about 1, with the implication that R_0, the radius of the elastoplastic interface for the reference hydrostatic case $m = 0$, represents the *average* radius of the elastoplastic boundary in the nonhydrostatic case ($m > 0$).

5.3. *Numerical results*

The non-linear system of equations was solved numerically using Brown's algorithm[11]. The computed coefficient $m_{2j}^{(n)}, j = 1, n$ and λ for various values of K are tabulated in Table 1 for the case $n = 5$, $m = 0.1$. Some solutions corresponding to a friction angle of 30° (both active and passive modes), obtained with $n = 5$, are illustrated in Fig. 3. As already suggested by the first-order solution, eqn (42), the interface has an oval shape with its great axis perpendicular to the major far-field compressive stress $P^0 + S^0$ in the active mode ($K = K_p$) but parallel to $P^0 + S^0$ in the passive mode ($K = K_a$). The eccentricity of the interface increases with the obliquity m.

Table 1. Coefficients of the mapping function ($n = 5$) for an obliquity $m = 0.1$

K	λ	$10 \cdot m_2$	$10^2 \cdot m_4$	$10^4 \cdot m_6$	$10^5 \cdot m_8$	$10^6 \cdot m_{10}$
2	0.9989	0.6689	−0.1125	0.5051	−0.2975	0.1997
3	0.9986	0.5025	−0.1269	0.6419	−0.4060	0.2868
4	0.9988	0.4024	−0.1220	0.6585	−0.4335	0.3147
5	0.9989	0.3356	−0.1131	0.6357	−0.4293	0.3174
1/2	1.0022	−1.3289	0.2221	0.4900	0.2027	0.1076
1/3	1.0037	−1.4926	0.3731	0.6135	0.2275	0.1130
1/4	1.0048	−1.5905	0.4763	0.6237	0.2156	0.1027
1/5	1.0055	−1.6558	0.5503	0.5986	0.1970	0.0913

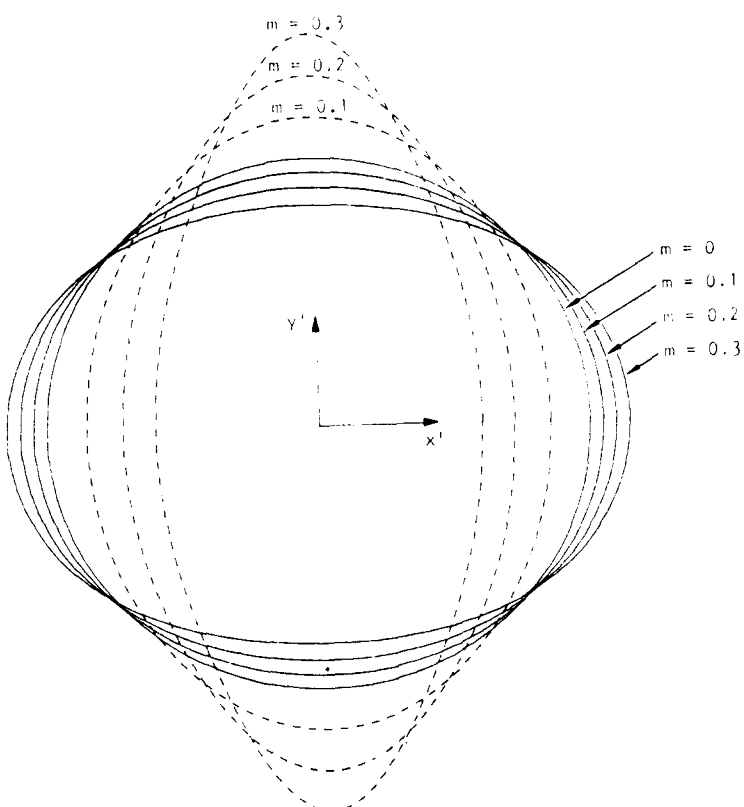

Fig. 3. Active (continuous line) and passive (dashed) solutions of the elastoplastic interface in the unit plane z', for a friction angle of 30°.

5.4. *Estimation of error*

Continuity of the mean pressure across Γ is automatically ensured if $\hat{\phi}_1(\zeta)$ is calculated from eqn (24). However, because of the approximate nature of the solution $\hat{\omega}^{(n)}(\zeta)$, continuity of the stress deviatoric across Γ will in general be violated. The jump of the stress deviatoric across the interface represents the error associated with the approximate solution.

Let the complex error $\hat{E}^{(n)}(\sigma)$ be defined as

$$\hat{E}^{(n)}(\sigma) := \left[\frac{\tau_{yy} - \tau_{xx}}{2} + \mathrm{i}\,\tau_{xy}\right] \Big/ S_1^0 \Big|_{\mathrm{e}}^{\mathrm{p}} \tag{44}$$

where p and e indicate that the stress deviatoric is evaluated respectively on the plastic and elastic side of Γ. The error $\hat{E}^{(n)}(\sigma)$ depends only on the parameters m and K. It can readily be verified, by following Laurent's decomposition for the second boundary condition, eqn (16), that $G^+(\zeta)$ is intimately related to the jump of the stress deviatoric across Γ. Taking into account, eqns (38) and (39), we have

$$\hat{E}^{(n)}(\sigma) = \left(\sum_{j=n}^{\infty} G_{2j}^{+}\sigma_{2j}\right) \Big/ \hat{\omega}'^{(n)}(\sigma) \tag{45}$$

The expression for the coefficients G_{2j}^{-} is derived in Appendix B. A global measure of the error associated with the approximation of order n can be defined as

$$\bar{E}^{(n)}(K,m) = \frac{2}{\pi}\int_0^{\pi/2} |\hat{E}^{(n)}(\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}\theta. \tag{46}$$

Table 2 illustrates the variation of $\bar{E}^{(n)}(K,m)$ with respect to the number of terms n, for a friction angle of 30°, and $m = 0.1$ and 0.3. An examination of this table suggests that the optimum number of terms n to be used for the approximation $\hat{\omega}^{(n)}(\zeta)$ is either 3 or 4.

Table 2. Variation of the error $\bar{E}$ with the number of terms in the mapping function and the obliquity (friction angle of 30°)

	Active mode ($K = 3$)		Passive mode ($K = 1/3$)	
n	$\bar{E}(m = 0.1)$	$\bar{E}(m = 0.3)$	$\bar{E}(m = 0.1)$	$\bar{E}(m = 0.3)$
1	2.56×10^{-3}	2.39×10^{-2}	2.51×10^{-3}	2.44×10^{-2}
2	7.68×10^{-5}	3.81×10^{-3}	2.46×10^{-5}	1.04×10^{-3}
3	5.83×10^{-5}	9.97×10^{-4}	1.54×10^{-5}	5.61×10^{-5}
4	4.97×10^{-5}	4.08×10^{-5}	1.69×10^{-5}	4.19×10^{-5}
5	5.03×10^{-5}	2.10×10^{-4}	1.69×10^{-5}	5.50×10^{-5}

6. DISCUSSION

The statical approach used to determine the position of the elastoplastic interface rests upon three assumptions; namely (1) the hole is completely engulfed by the plastic zone, (2) the problem is statically determinate and (3) no elastic unloading takes place during development of the plastic zone. In the sequel, the limiting conditions for which the constructed solution does not violate assumptions (1) and (2), and a loading path conjectured to be consistent with assumption (3) are discussed.

6.1. *Loading path*

Providing the stress at infinity is characterized by an obliquity m less than 1/2, there exists an interval (p_{ea}, p_{ep}) for the internal pressure p, for which the stress field calculated according to the Kirsch elastic solution (e.g. Ref. [12]) does not violate the yield criterion (3).

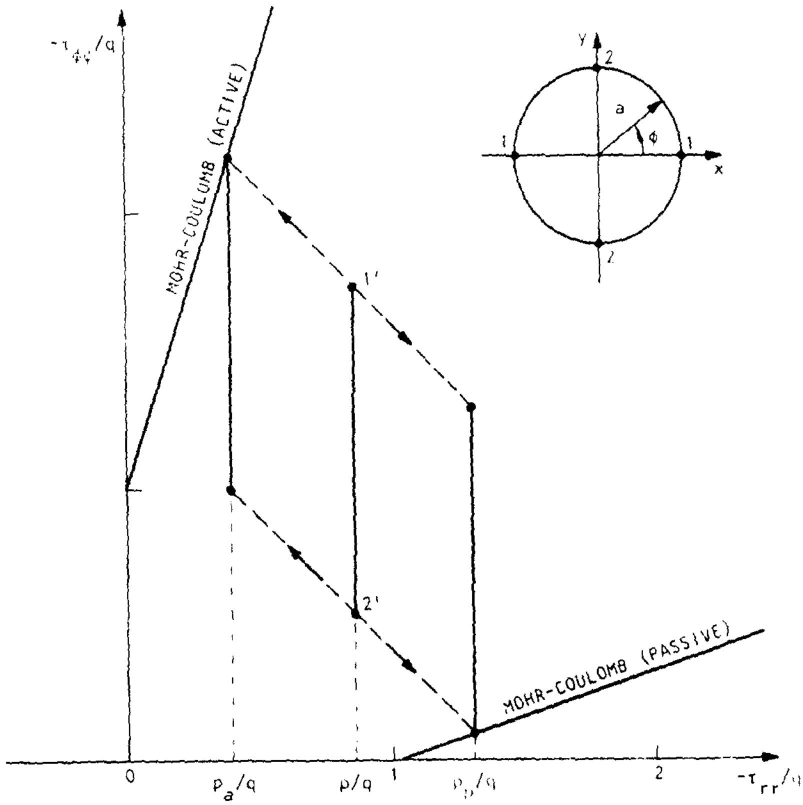

Fig. 4. Elastic stress profile at the boundary.

The two elastic limits p_{ea} and $p_{ep}(p_{ea} \leqslant p_{ep})$ are given by the general formula

$$p_e = \pm \frac{2S_i^0}{K-1}\left(1 + 2\frac{K-1}{K+1}m\right) - \frac{q}{K_p - 1} \tag{47}$$

where the upper sign and $K = K_p$ apply for the active elastic limit p_{ea}, the lower sign and $K = K_a$ for the passive elastic limit p_{ep}.

Figure 4 illustrates the elastic stress profile on the boundary $r = a$, at the two elastic limits and at some pressure p within the elastic interval (p_{ea}, p_{ep}). There is impending failure in the active mode for $p = p_{ea}$, at the boundary points designated 1 in Fig. 4 ($\phi = 0, \pi$); in the passive mode for $p = p_{ep}$ at points 2 ($\phi = -\pi/2, \pi/2$).

Hence, starting from an elastic state corresponding to a stress τ^0 at infinity (characterized by an obliquity less than 1/2) and an internal pressure p in the interval (p_{ea}, p_{ep}), a monotonic increase of p beyond the elastic limit p_{ep} will cause the propagation around points 2 of yield zones in passive limit equilibrium. Similarly, a decrease of p beyond the elastic limit p_{ea}, will induce the formation of plastic regions in active limit equilibrium around points 1. To be consistent with these loading paths, the statical solution of Γ should correspond to internal pressure $p \geqslant p_{ep}$ in the passive mode, but $p \leqslant p_{ea}$ in the active one. Calculation of the internal pressure corresponding to the first admissible configuration of the interface is carried out in the next article.

It is to be noted that a complete load history bringing the system from a uniform stress τ^0 in the plane to a stress-free boundary $r = a$ has an important application for the analysis of excavation induced failure of rock around deep tunnels[13, 14].

6.2. *Tangency of the interface to the hole*

The first admissible configuration of the interface corresponds to Γ being tangent to the hole boundary. From the results of Section 5, the points of tangency are located at

$Z = \pm i\,a$ in the active mode ($K = K_p$) and at $Z = \pm a$ in the passive mode ($K = K_a$). The tangency condition can therefore be translated as

$$\lambda R_0\left(1 + \sum_{j=1}^{\infty} (\mp 1)^j m_{2j}\right) = 1. \tag{48}$$

In regard to the loading path discussed above, the tangency condition (48) is transformed into an equation giving the minimum internal pressures p_l—at which the elastoplastic boundary completely encloses the hole—as a function of S_l^0, m, q, and K. Using eqns (5), (11) and (12), it can be deduced from eqn (48) that

$$p_l = \pm \frac{2S_l^0}{K-1}\lambda^{K-1}\left(1 + \sum_{j=1}^{\infty} (\mp 1)^j m_{2j}\right)^{K-1} - \frac{q}{K_p - 1}. \tag{49}$$

The variation of $\hat{p}_e$ and $\hat{p}_l$, where $\hat{p}$ is defined as

$$\hat{p} = \left(p + \frac{q}{K_p - 1}\right)\bigg/ S_l^0 \tag{50}$$

with the obliquity m is tabulated in Table 3 for a friction angle of 30° (both passive and active modes). It is apparent from this table that indeed $p_{la} \leqslant p_{ea}$ and $p_{lp} \geqslant p_{ep}$. Furthermore, since R_0 decreases monotonically with p if $K = K_p$ but increases with p if $K = K_a$, the statical solution is applicable for $p \leqslant p_l$ in the active mode but $p \geqslant p_l$ in the passive mode.

Table 3. Variation of $\hat{p}_e$ and $\hat{p}_l$ with the obliquity m for both modes of failure (friction angle of 30°)

	Active mode ($K = 3$)		Passive mode ($K = 1/3$)	
m	$\hat{p}_e$	$\hat{p}_l$	$\hat{p}_e$	$\hat{p}_l$
0.0	1.000	1.000	3.000	3.000
0.1	1.100	0.897	2.700	3.323
0.2	1.200	0.787	2.400	3.695
0.3	1.300	0.667	2.100	4.118

Since the shape of Γ is independent of p, the interface propagates outwards in a self-similar manner as p increases monotonically beyond p_{lp}, or decreases from p_{la}. This self-similarity property implies that the statical solution is consistent with the assumption of no elastic unloading for the particular load path discussed above (provided that the first admissible configuration is indeed reached).

6.3. *Tangency to a stress characteristic*

The elastoplastic interface is statically determinate if any point on Γ can be connected to the boundary $r = a$, by two stress characteristics (or slip line) lying entirely within the plastic region. At the limit of statical determinacy, the interface becomes tangent to a stress characteristic (see Fig. 5). In this particular problem, the slip lines consist of logarithmic spirals inclined about the radial direction by an angle α' equal to $(\pi/4 + \Phi/2)$ in the active mode, and to $(\pi/4 - \Phi/2)$ in the passive mode. Hence, if the interface is tangent to a stress characteristic, the angle $\alpha - \phi$ at the point of tangency (where α is the angle between the outward normal to the interface and the x-axis and ϕ is the cylindrical coordinate angle) is given by

$$|\alpha - \phi| = \frac{\pi}{2} - \alpha'. \tag{51}$$

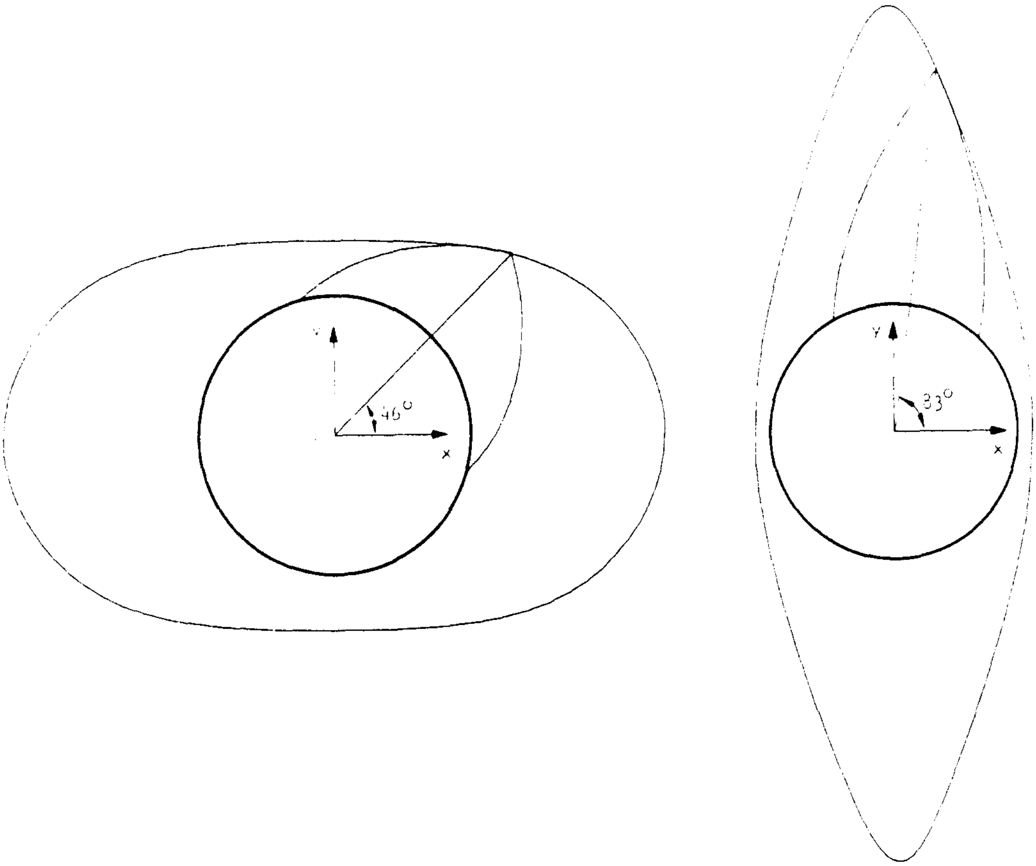

Fig. 5. Limit of statical determinacy for active (left) and passive (right) mode of failure (friction angle of 30°).

The condition of tangency between the interface and one of the stress characteristics is controlled solely by the two parameters m and K; indeed, the shape of the slip line depends on K only, the shape of Γ on m and K. The limiting condition regarding statical determinacy of the problem may therefore be translated as a limiting value $m_*(K)$ of the obliquity m, the problem being statically determinate if

$$0 \leqslant m \leqslant m_*(K). \tag{52}$$

The maximum obliquity $m_*(K)$ is computed by requiring that $\alpha - \phi = \pm\pi/4 - \Phi/2$ at the point on the interface at which $\alpha - \phi$ is extremum. These two conditions translate as

$$\sigma_*^2 \frac{\hat{\omega}'(\sigma_*)}{\overline{\hat{\omega}'(\sigma_*)}} \frac{\overline{\hat{\omega}(\sigma_*)}}{\hat{\omega}(\sigma_*)} = \pm i e^{-i\Phi} \tag{53}$$

$$1 + \mathrm{Re}\left[\sigma_* \frac{\hat{\omega}''(\sigma_*)}{\hat{\omega}'(\sigma_*)} - \sigma_* \frac{\hat{\omega}'(\sigma_*)}{\hat{\omega}(\sigma_*)}\right] = 0. \tag{54}$$

Equations (53) and (54) constitute a system of equations, parametric in K, to be solved for m_* and σ_*. Here σ_* is the image in the reference plane of the point on Γ at which $\alpha - \phi$ is extremum.

The system of eqns (41), with the obliquity m treated as the unknown m_*, was solved simultaneously with eqns (53) and (54) for the $(n + 2)$ unknowns $m_{2j}^{(n)}$, $j = 1, n$, m_*, and σ_*, for various friction angles in the active and passive modes. The results of this investigation are summarized in Table 4, where the limiting obliquity m_* is given as a function of the friction angle for both modes of failure (case $n = 3$).

For a frictionless material, $m_* = 2^{1/2} - 1$ (both modes); this value is obtained by solving analytically the system of eqns (53) and (54), using the closed-form solution, eqn $(35)_2$. Note that both Hill[6] and Savin[2] quote a limiting value $m_* = (2^{1/2} - 1)/(2^{1/2} + 1)$ for the case $K = 1$. This value, which is stated without any supporting analysis, is incorrect.

Table 4. Limiting obliquity m_* ($n = 3$)

Friction angle Φ	m_* (active)	m_* (passive)
10°	0.4197	0.4047
20°	0.4195	0.3955
30°	0.4200	0.3879
40°	0.4263	0.3819
50°	0.4410	0.3773

7. CONCLUSIONS

An approximate statical solution of the elastoplastic interface has been derived for the problem of Galin with a cohesive-frictional material, for configurations where the hole is completely engulfed by a plastic zone. Like in Galin's solution, the location of the elastoplastic interface is solved as a problem of conformal mapping (by making use of the Kolossov–Muskhelishvili functions), based on the assumptions that (1) the interface is statically determinate and (2) the plastic region has developed without any elastic unloading.

The method of solution presented in this paper leads to the formulation of a scalar condition and a functional equation for determination of the mapping function. Although the functional equation can be solved in closed form for a Tresca material (Galin's solution), it has to be solved approximately in the general case of a Mohr–Coulomb material, by seeking a solution to the mapping function in the form of a truncated series expansion. The coefficients of the truncated series expansion are shown to be the roots of a non-linear system of algebraic equations.

Calculation indicates that the elastoplastic interface has an oval shape with its major axis oriented parallel or perpendicular to the greatest compressive stress at infinity depending on whether the stress in the plastic zone is in passive or active limit equilibrium. The shape of the elastoplastic boundary is solely controlled by the obliquity m of the stress at infinity and by the friction angle of the material. The average size of the plastic zone was also shown to be given by the hydrostatic solution.

Requirements for consistency of the solution were discussed. It has been proven that a statical solution of Γ can be constructed provided that the obliquity is less than a critical value m_*, function of the friction angle and the mode of limit equilibrium (passive or active). The assumption of no elastic unloading was examined in conjunction with a particular loading path, where the magnitude of the internal pressure is monotonically varied. For this particular load path, it was shown that a continuous growth of the plastic region is predicted.

An important application of this solution is in the assessment of the extent of failed rock around a deep tunnel subject to a non-hydrostatic *in situ* stress and in the evaluation of the excavation-induced closure of the tunnel[13, 14].

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APPENDIX A
A SUFFICIENT CONDITION FOR THE INTERMEDIACY OF THE OUT-OF-PLANE STRESS

It is proven that

$$\nu \geqslant \frac{1}{(K_p + 1)} \tag{A1}$$

represents a sufficient condition for τ_{zz} to be the strict intermediate principal stress τ_2 in regions of plastic flow, provided that the stress at infinity satisfies the plane strain condition

$$\tau^0_{zz} = \nu(\tau^0_{xx} + \tau^0_{yy}). \tag{A2}$$

The proof relies on the existence of a potential function H for the plastic strain increment $\mathrm{d}\varepsilon^p_{ij}$

$$\mathrm{d}\varepsilon^p_{ij} = \mu\frac{\partial H}{\partial \tau_{ij}}, \qquad \mu \geqslant 0 \tag{A3}$$

and on the assumption that H is a function of stress of the same form as the yield function. In terms of the major and minor principal stresses τ_1, τ_3, the expression for H is then

$$H = K^*_p\tau_1 - \tau_3 - q^* \tag{A4}$$

where K^*_p is the dilatancy factor $(1 \leqslant K^*_p \leqslant K_p)$ and q^* is an arbitrary constant. It follows from eqns (A3) and (A4) that the condition for the plane criterion (3) to be equivalent to the general Mohr–Coulomb criterion (1) (i.e. τ_1, τ_3 in the plane of deformation) is, in effect, identical to the condition of plastic flow in the plane of deformation ($\mathrm{d}\varepsilon^p_{zz} = 0$).

If at any point where the yield criterion $F = 0$ is satisfied, the out-of-plane stress τ_{zz} has been the strict intermediate principal stress τ_2 since the onset of plastic deformation, the plane strain condition $\varepsilon_{zz} = 0$ is equivalent to

$$\varepsilon^e_{zz} = \varepsilon^p_{zz} = 0 \tag{A5}$$

and, therefore

$$\tau_{zz} = \nu(\tau_{xx} + \tau_{yy}) \tag{A6}$$

everywhere in the plane of deformation. It can readily be proven that τ_{zz}, calculated from eqn (A6), is the intermediate principal stress τ_2 (strict) if

$$S \geqslant (1 - 2\nu)P, \qquad P \geqslant 0. \tag{A7}$$

Hence, providing that (Fig. A1)

$$(1 - 2\nu) \leqslant \sin\Phi = \frac{K_p - 1}{K_p + 1} \tag{A8}$$

$\tau_{zz} = \tau_2$ when $S = S_1$ and $P > 0$. (For cohesionless material, inequality (A8) must be strict.) It can also be verified that the yield condition $F = 0$ is never reached for $0 \leqq S \leqq (1 - 2\nu)P$, provided that inequality (A8) is met. Hence, inequality (A1) or (A8) represents a sufficient condition for $S = S_1$ to be equivalent to $F = 0$.

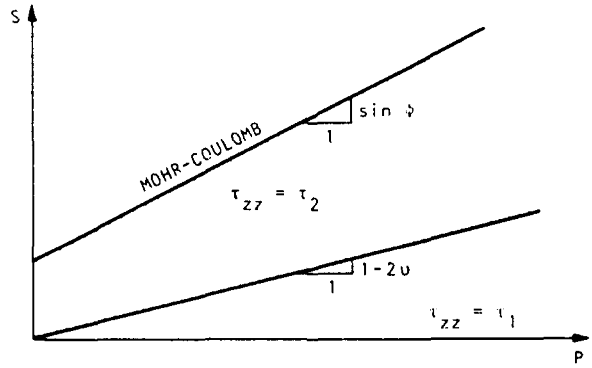

Fig. A1. Graphical representation of the conditions for which τ_{zz} is the intermediate principal stress τ_2.

APPENDIX B
EQUATIONS FOR THE COEFFICIENTS OF THE MAPPING FUNCTION

In this appendix, the superscript (n)—used to denote reference to the approximation of order n of the mapping function—is dropped for simplicity of notation.

Expression for the coefficient λ

The expression of λ in terms of the coefficients m_{2j}, $j = 1, n$ is deduced from constraint (26), which imposes that the zero-order coefficient of the Laurent series expansion of $f_*^{K-1}(\zeta)$ be equal to 1. The Laurent series expansion of $f_*^{K-1}(\zeta)$ needs first to be derived. For that purpose, the function $f_*^{K-1}(\zeta)$ is rewritten as

$$f_*^{K-1}(\zeta) = \lambda^{K-1} S^{\kappa}(\zeta)\bar{S}^{\kappa}(\zeta^{-1}) \tag{B1}$$

where

$$S(\zeta) = \frac{\dot{\omega}(\zeta)}{\lambda\zeta} = 1 + \sum_{j=1}^{n} m_{2j}\zeta^{-2j} \tag{B2}$$

$$\kappa = \frac{(K - 1)}{2}. \tag{B3}$$

Let $\rho_0(<1)$ be the largest of the moduli of the zeros of $\dot{\omega}(\zeta)$. Because of the symmetry properties, eqns (8), of the mapping function, $S(\zeta)$ transforms the infinite region Σ_0^- $(\rho > \rho_0)$ onto a finite domain D which lies on the right of the imaginary axis $\mathrm{Re}[S(\zeta)] = 0$. The function $S^{\kappa}(\zeta)$ is therefore single-valued for $\zeta \in \Sigma_0^-$, if the negative real axis of the complex plane $S(\zeta)$ is selected as branch cut. The Laurent series expansion about $\zeta = 0$ of $S^{\kappa}(\zeta)$ in Σ_0^- is then given by

$$S^{\kappa}(\zeta) = \sum_{j=0}^{\infty} c_{2j}\zeta^{-2j}, \qquad \zeta \in \Sigma_0^-. \tag{B4}$$

The coefficients c_{2j}, which are real, are calculated using the recursive formula of Miller[15]

$$c_0 = 1$$
$$c_{2j} = \frac{1}{j}\sum_{l=1}^{l_*} [(\kappa + 1)l - j]c_{2(j-l)}m_{2l}; \qquad j = 1, \infty \tag{B5}$$

with

$$l_* = \begin{cases} j; & j = 1, n \\ n; & j = n, \infty \end{cases}$$

Note that if κ is a positive integer, the coefficients c_{2j} are identically zero for $j > n\kappa$.

The Taylor expansion about $\zeta = 0$ of the associated function $\bar{S}^{\kappa}(\zeta^{-1})$ is immediately deduced from eqn (B4)

$$\bar{S}^{\kappa}(\zeta^{-1}) = \sum_{j=0}^{\infty} c_{2j}\zeta^{2j}, \qquad \zeta \in \Sigma_0^+. \tag{B6}$$

The Cauchy product of the two series (B4) and (B6) converges absolutely and uniformly to $[f_\bullet(\zeta)/\lambda]^{\kappa-1}$ in the ring Ω_0 ($\rho_0 < \rho < \rho_0^{-1}$)

$$f_\bullet^{\kappa-1}(\zeta) = \lambda^{\kappa-1}\left[\sum_{j=1}^{\infty} d_{2j}\zeta^{2j} + d_0 + \sum_{j=1}^{\infty} d_{2j}\zeta^{-2j}\right], \qquad \zeta \in \Omega_0 \tag{B7}$$

where

$$d_{2j} = \sum_{k=0}^{\infty} c_{2k}c_{2(k+j)}, \qquad j = 0, \infty. \tag{B8}$$

In light of eqn (B7), the function $d(\zeta)$ resulting from the Laurent decomposition (20) of $f_\bullet^{\kappa-1}(\zeta)$ reads

$$d(\zeta) = \lambda^{\kappa-1}\sum_{j=1}^{\infty} d_{2j}\zeta^{-2j}, \qquad \zeta \in \Sigma_0^- \tag{B9}$$

and constraint (26) becomes

$$\lambda = d_0^{1/(1-\kappa)}. \tag{B10}$$

The above equation gives λ in terms of the coefficients m_{2j}, $j = 1, n$ of the mapping function.

Systems of equations for m_{2j}, $j = 1$, n

The equations needed to calculate the coefficients m_{2j}, $j = 1$, n are determined by imposing the vanishing of the n first coefficients of Taylor expansion (38) of $G^+(\zeta)$. According to eqn $(32)_2$, the function $G^+(\zeta)$ is defined in terms of the analytic functions $g_1(\zeta)$ and $h_1(\zeta)$ resulting from the Laurent decomposition of $g(\zeta)$ and $h(\zeta)$. In the sequel, the Taylor series of $g_1(\zeta)$ and $h_1(\zeta)$ is directly deduced from the Laurent series expansion of $g(\zeta)$ and $h(\zeta)$.

First, consider the function $g(\zeta)$ defined in eqn (28). The Laurent series expansion of $g(\zeta)$ is calculated by Cauchy product of the series representation of $\bar{\omega}(\zeta^{-1})$ and $d'(\zeta)$. Since

$$\bar{\omega}(\zeta^{-1}) = \lambda\zeta^{-1}\left(1 + \sum_{j=1}^{n} m_{2j}\zeta^{2j}\right) \tag{B11}$$

$$d'(\zeta) = -\lambda^{\kappa-1}\zeta^{-1}\sum_{j=1}^{\infty} 2j\, d_{2j}\zeta^{-2j} \tag{B12}$$

we have that

$$g(\zeta) = \frac{K_p+1}{K_p-1}\lambda^{\kappa}\sum_{j=0}^{n-2} g'_{2j}\zeta^{2j} + \frac{K_p+1}{K_p-1}\lambda^{\kappa}\sum_{j=1}^{\infty} g''_{2j}\zeta^{-2j}, \qquad \zeta \in \Omega_0. \tag{B13}$$

The first series in eqn (B13) identifies $g_1(\zeta)$, the second $g_2(\zeta)$. Hence

$$g_1(\zeta) = \frac{K_p+1}{K_p-1}\lambda^{\kappa}\sum_{j=0}^{n-2} g'_{2j}\zeta^{2j} \tag{B14}$$

where

$$g'_{2j} = \sum_{k=1}^{n-j-1} 2k\, d_{2k}m_{2(k+j+1)}, \qquad j = 0,\ n-2. \tag{B15}$$

(Note that $g_1(\zeta) = 0$ if $n = 1$.)

Consider next the function $h(\zeta)$, defined in eqn (30). $h(\zeta)$ can advantageously be rewritten in terms of $S(\zeta)$, defined in eqn (B2), and its associated function $\bar{S}(\zeta^{-1})$

$$h(\zeta) = \lambda^{\kappa-1}\zeta^{-2}\omega'(\zeta)[\bar{S}(\zeta^{-1})]^{(\kappa+1)/2}[S(\zeta)]^{(\kappa-3)/2}. \tag{B16}$$

The Taylor series of $[\bar{S}(\zeta^{-1})]^{(\kappa+1)/2}$ about $\zeta = 0$ is

$$[\bar{S}(\zeta^{-1})]^{(\kappa+1)/2} = \sum_{j=0}^{\infty} c'_{2j}\zeta^{2j}, \qquad \zeta \in \Sigma_0^+ \tag{B17}$$

and the Laurent series expansion of $[S(\zeta)]^{(K-3)/2}$ with respect to $\zeta = 0$ is

$$[S(\zeta)]^{(K-3)/2} = \sum_{j=0}^{\infty} c''_{2j}\zeta^{-2j}, \qquad \zeta \in \Sigma_0^- . \tag{B18}$$

The real coefficients c'_{2j} and c''_{2j} are calculated by means of the recursive formula (B5) with κ replaced by $(K+1)/2$ and by $(K-3)/2$, respectively. The Cauchy product of the two series (B17) and (B18) converges for $\zeta \in \Omega_0$

$$[\bar{S}(\zeta^{-1})]^{(K+1)/2}[S(\zeta)]^{(K-3)/2} = \sum_{j=-\infty}^{\infty} e_{2j}\zeta^{2j} \tag{B19}$$

with

$$e_{2j} = \sum_{k=0}^{\infty} c'_{2(j+k)}c''_{2k}, \qquad j = 0, \infty$$
$$e_{-2j} = \sum_{k=0}^{\infty} c'_{2k}c''_{2(j+k)}, \qquad j = 1, \infty. \tag{B20}$$

The Laurent expansion of $h(\zeta)$ is then obtained by Cauchy product of eqn (B19) with the polynomial $\bar{\omega}'(\zeta)$ ζ^{-2}. Partial summation of the terms with zero and positive powers of ζ in that series identifies the function $h_1(\zeta)$ (same procedure as for $g_1(\zeta)$). All calculations done

$$h_1(\zeta) = \lambda^K \sum_{j=0}^{\infty} h'_{2j}\zeta^{2j} \tag{B21}$$

with

$$h'_{2j} = e_{2(j+1)} - \sum_{k=1}^{n} (2k-1)m_{2k}e_{2(k+j+1)}. \tag{B22}$$

It follows from the definition of eqn $(32)_2$ of $G^+(\zeta)$ and from eqns (B14) and (B21) that the coefficients G^+_{2j} of the Taylor expansion (38) of $G^+(\zeta)$ are

if $n > 2$

$$G_0^+ = \pm\lambda^K\frac{K+1}{K-1}g'_0 \pm \lambda^K h'_0 - m\lambda$$
$$G^+_{2j} = \pm\lambda^K\frac{K+1}{K-1}g'_{2j} \pm \lambda^K h'_{2j}, \qquad j = 1, n-2$$
$$G^+_{2j} = \pm\lambda^K h'_{2j}, \qquad j = n-1, \infty \tag{B23}$$

if $n = 1$ or 2

$$G_0^+ = \pm\lambda^K\frac{K+1}{K-1}g'_0 \pm \lambda^K h'_0 - m\lambda$$
$$G^+_{2j} = \lambda^K h'_{2j}, \qquad j = 1, \infty$$
$$(g'_0 = 0 \text{ if } n = 1). \tag{B23'}$$

The n equations needed for the determination of the coefficients m_{2j}, $j = 1, n$ are obtained by imposing the vanishing of G^+_{2j}, $j = 0, n-1$. After dividing G^+_{2j} by λ^K and making use of eqn (B10), we have

if $n > 2$

$$\frac{K+1}{K-1}g'_0 + h'_0 \mp md_0 = 0$$
$$\frac{K+1}{K-1}g'_{2j} + h'_{2j} = 0, \qquad j = 1, n-2$$
$$h'_{2(n-1)} = 0 \tag{B24}$$

if $n = 2$

$$\frac{K+1}{K-1}g'_0 + h'_0 \mp md_0 = 0$$
$$h'_2 = 0 \tag{B24'}$$

if $n = 1$

$$h'_0 \mp md_0 = 0. \tag{B24''}$$

Equations (B24) constitute a system of n non-linear equations in the coefficients m_{2j}, $j = 1, n$ of the mapping function (λ is absent from eqns (B24)).

First-order approximation of interface

An approximate explicit solution of the interface can be found by linearizing eqn (B24″). Using eqns (B22), eqn (B24″) becomes

$$e_2 - m_2 e_4 \mp m d_0 = 0. \tag{B25}$$

A first-order estimate of the coefficient m_2 can then be obtained by expressing e_2, e_4, and d_0 in terms of m_2 with the help of eqns (B8) and (B20), and linearizing the resulting equation in m_2

$$m_2 \cong \pm \frac{2}{K+1} m. \tag{B26}$$

An approximate expression of the coefficient λ can also be determined from eqn (B10), using

$$d_0 \cong 1 + c_2^2 = 1 + \frac{K-1}{2} m_2. \tag{B27}$$

Hence

$$\lambda \cong \left[1 + \left(\frac{K-1}{K+1} m\right)^2\right]^{1/(1-K)}. \tag{B28}$$